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Derivation of Schrödinger's Equation from Stochastic Electrodynamics

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Abstract

Using exclusively classical arguments, an equation similar to Schrödinger's equation is derived within the framework of stochastic electrodynamics.

This equation, when compared to Schrödinger's, has two extra terms. It is shown that, for a harmonic oscillator, these terms are negligible as compared to the ground-state average energy. Moreover, with a good approximation, they cancel each other.

1. *Introduction*

As a consequence of Wheeler and Feynman's absorber theory of radiation (Wheeler & Feynman, 1945), Braffort *et al.* (1954) have considered the existence, at the absolute zero of temperature, of a classical fluctuating electromagnetic field. Application of the Lorentz-invariance to its spectrum led Braffort & Tzara (1954), and, more explicitly, Marshall (1963, 1965) and Boyer (1969) to derive, for a one-dimensional case, the following expression of the spectral density of this zero-point field

$$\epsilon(\omega) = \frac{K\omega^3}{3\pi c^3} \tag{1.1}$$

where K is a constant, having the dimension of action.

Braffort & Tzara (1954) and Braffort *et al.* (1965) have considered the effect of the zero-point fluctuating electromagnetic field on a harmonic oscillator. For the non-relativistic case, where $e.\dot{\mathbf{x}} \wedge \mathbf{B}(t)$ may be neglected, the equation of motion of the harmonic oscillator is

$$-\frac{2e^2}{3c^3}.\dddot{x} + m.\ddot{x} + a.x = eE(t) \tag{1.2}$$

$E(t)$ is the fluctuating zero-point electric field.

If one considers equation (1.2) at time t and time $t+\tau$, and then multiplies the left members together and the right members together and takes the average over an infinite interval of time t, one obtains an equation relating the autocorrelation functions of x and the autocorrelation function of $E(t)$. Taking the Fourier transform of both sides of this equation, the Fourier transform of the right-hand side is precisely e^2 times the spectral density given by equation (1.1), and one may obtain expressions of the average kinetic and potential energies. Finally, the average total energy is given by

$$\langle E_{\text{tot}}\rangle = \frac{K\omega_0}{2}\left[1 - \frac{1}{2\pi}\frac{\omega_0}{\omega_s}\log\frac{\omega_0}{\omega_s}\right] \tag{1.3}$$

where $\omega_0{}^2 = a/m$ and $\omega_s = 3mc^3/2e^2$.

Braffort & Taroni (1967) have considered the action of the zero fluctuating field on a free electron, moving in a uniform magnetic field H. They found that the average total energy of the electron is given by

$$\langle E_{\text{tot}}\rangle = \frac{K\omega_H}{2}\left[1-\frac{\pi}{2}\left(\frac{\omega_H}{\omega_s}+\log\frac{\omega_H}{\omega_s}\right)\right] \tag{1.4}$$

where $\omega_H = eH/mc$.

Marshall (1963), Surdin *et al.* (1966) and Boyer (1969), within the framework of stochastic electrodynamics, have obtained Planck's law of black-body radiation, where

$$\langle E(\omega, T)_{\text{tot}}\rangle = K\omega\left[\frac{1}{\exp(K\omega/kT)-1}+\frac{1}{2}\right] \tag{1.5}$$

Marshall (1963) and Surdin (1970) have considered the probability distribution for a harmonic oscillator and for a free electron moving in a uniform magnetic field. Assuming that the action of the fluctuating zero-point field is a Markoff process, Chandrasekhar's method (Chandrasekhar, 1943) was applied in the case where $\omega_0/\omega_s \ll 1$.

Using this method, the probability distributions and, in the stationary state, the average total energy, are readily obtained. For the harmonic oscillator equation (1.3), and for the free electron in a uniform magnetic field equation (1.4), were obtained by Surdin (1970).

It was also shown (Surdin, 1970) that, although the spectral density goes to infinity as ω^3, the harmonic oscillator and the free electron in a uniform magnetic field, due to the form of their band-pass, absorb a finite energy from the zero-point field. This remark explains why in stochastic electrodynamics one dispenses with subtracting and cut-off procedures.

If, according to Heisenberg (Heisenberg, 1927; Jammer, 1966), one defines the uncertainty of the position δq and the uncertainty of the momentum δp, one obtains (Surdin, 1970) for a harmonic oscillator and for the free electron in a uniform magnetic field

$$\delta p\,.\,\delta q = K \tag{1.6}$$

Thus, using exclusively classical arguments, the results obtained in stochastic electrodynamics would be the same as those obtained in quantum electrodynamics if one wrote

$$K \equiv \hbar \tag{1.7}$$

As stated above, K is a constant obtained by purely classical considerations, i.e. from Lorentz-invariance of the spectrum of the zero-point fluctuating electromagnetic field.

The derivation, within the framework of stochastic electrodynamics, of Schrödinger's equation would achieve, if not a complete identification, at least a parallel between wave mechanics and stochastic electrodynamics. The remainder of this article will be devoted to the derivation of Schrödinger's equation.

2. *Preliminary Remarks*

A formal analogy between the differential equation giving the probability $\rho(x,t)$, for a system in Brownian motion involving a Markoff process and the Schrödinger equation was noticed by Métadier (1931), Fürth (1933) and developed recently by Comisar (1965).

The analogy between $\rho(x,t)$ and Schrödinger's wave function ψ is inconsistent, mainly because

(1) the probability density for a Brownian motion is given by $\rho(x,t)$ and according to this analogy should be given by ψ in wave mechanics, whereas, in fact, it is $\psi.\psi^*$;

(2) to complete the analogy one has to consider an imaginary diffusion coefficient.

Later work by Kershaw (1964), Nelson (1966), de la Peña-Auerbach (1967) and Boyer (1968) has improved the analogy by considering that ψ is analogue to $\sqrt{[\rho(x,t)]}$. The analogy requires the diffusion coefficient D to be real and equal to $D=\hbar/2m$. In other words, a classical system in Brownian motion is subject to random fluctuations in position, whose scale is determined by $\hbar/2m$. Writing $D=\hbar/2m$ constitutes an entirely independent postulate, which appears in the same way in quantum mechanics (de la Peña-Auerbach, 1967).

A critical analysis of some of the above-mentioned papers was made by Gilson (1968). His main conclusion is that the only situation where stochastic theory and the Schrödinger equation could be consistent, for a real potential function, is when the coefficient of diffusion is identically zero.

The derivation of Schrödinger's equation given hereafter is considered to be free from the objection raised by Gilson. It is based entirely on classical considerations and obtained by a systematic use of the results of stochastic electrodynamics referred to in the introduction. The actual mathematical approach follows closely that given by Olbert in Hayakawa's paper (Hayakawa, 1965).

3. *The Derivation of Schrödinger's Equation*

Consider the one-dimensional generalised Langevin equation

$$\ddot{x}+\beta\dot{x}-\frac{1}{m}F(x,t)=A(t) \tag{3.1}$$

where $F(x,t)$ is an external field of force and $A(t)$ is the fluctuating field whose action on the system is considered to be a Markoff process.

Marshall (1963) and Surdin (1970) have shown how equation (3.1) may be obtained from the equation of motion of the harmonic oscillator [equation (1.2)] when $\omega_0/\omega_s \ll 1$.

The Fokker–Planck equation giving the probability distribution $f(x,p,t)$, with $p=m.\dot{x}$ is then (Chandrasekhar, 1943)

$$\frac{\partial f}{\partial t}+\frac{p}{m}\frac{\partial f}{\partial x}+F(x,t)\frac{\partial f}{\partial p}=\beta\frac{\partial}{\partial p}(f.p)+\frac{Sm^2}{2}\frac{\partial^2 f}{\partial p^2} \tag{3.2}$$

where

$$S = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left\{ \int_t^{t+\Delta t} A(t)\,dt \right\}^2 \right\rangle \tag{3.3}$$

The evaluation of S in the case of a harmonic oscillator and in the case of a free electron in a uniform magnetic field was given by Surdin (1970), where use was made of equation (1.1).

Consider now the Fourier transform of the distribution function $f(x,p,t)$ given by

$$\rho(x,\xi,t) = \int_{-\infty}^{+\infty} f(x,p,t) \exp(2i\xi p/K)\,dp \tag{3.4}$$

in view of equation (1.6) this is a licit operation.

Using equation (3.4), equation (3.2) becomes

$$\frac{\partial f}{\partial t} - \frac{i}{2}\frac{K}{m}\frac{\partial^2 \rho}{\partial\xi\partial x} - \frac{2i\xi}{K}F(x,t)\,\rho = -\beta\xi\frac{\partial\rho}{\partial\xi} - \frac{2Sm^2}{K^2}\xi^2\rho \tag{3.5}$$

Taking a new set of variables, namely $r = x + \xi$, $r' = x - \xi$, equation (3.5) becomes

$$\frac{\partial f}{\partial t} - \frac{iK}{2m}\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial r'^2}\right)\rho - \frac{i}{K}(r - r')F\rho = -\beta\frac{r - r'}{2}\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial r'}\right)\rho -$$
$$- \frac{Sm^2}{2K^2}(r - r')^2\rho \tag{3.6}$$

Using the mean value theorem, one has

$$(r - r')F\left(\frac{r + r'}{2}\right) = \int_{r'}^{r} F(u)\,du = -[V(r) - V(r')] \tag{3.7}$$

where V is the potential function of the external field of force. One also has

$$\rho(r) - \rho(r') - (r - r')\rho'(r) = \frac{r - r'}{2}\left(\frac{\partial}{\partial r} - \frac{\partial}{\partial r'}\right)\rho \tag{3.8}$$

Consider a function $I(r)$ such as $I''(r) = \rho(r)$, then

$$I(r) - I(r') - (r - r')I'(r) = \tfrac{1}{2}(r - r')^2\rho \tag{3.9}$$

Using equations (3.7), (3.8) and (3.9), equation (3.6) becomes

$$\frac{\partial\rho}{\partial t} - \frac{iK}{2m}\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial r'^2}\right)\rho + \frac{i}{K}[V(r) - V(r')]\rho$$
$$= -\beta[\rho(r) - \rho(r')] + \beta(r - r')\rho'(r) - \frac{Sm^2}{K^2}[I(r) - I(r')] + \frac{Sm^2}{K^2}(r - r')I'(r) \tag{3.10}$$

One considers now a function ψ, such as

$$\rho(r,r',t)=\psi(r,t)\,.\,\psi^*(r',t) \tag{3.11}$$

Equation (3.10) may then be decomposed into

$$\begin{aligned}\frac{\partial\psi}{\partial t}-\frac{iK}{2m}\frac{\partial^2\psi}{\partial r^2}+\frac{i}{K}V(r,t)\psi=-\beta\psi+\frac{\beta}{2}(r-r')\,\rho'(r)-\\-\frac{Sm^2}{K^2}I(r)+\frac{Sm^2}{K^2}(r-r')I'(r)\end{aligned} \tag{3.12}$$

and a similar equation for ψ^* with $I_\psi''(r)=\psi$.

For $r=r'$ one has

$$\frac{\partial\psi}{\partial t}-\frac{iK}{2m}\frac{\partial^2\psi}{\partial r^2}+\frac{i}{K}V(r,t)\psi=-\beta\psi-\frac{Sm^2}{K^2}I_\psi(r,t) \tag{3.13}$$

and its adjoint equation.

When $\beta\equiv 0$ and $S\equiv 0$ the similarity between equation (3.13) and Schrödinger's equation is striking. To achieve complete identification (when $\beta\equiv 0$, $S\equiv 0$) one has to use equation (1.7), i.e. $K=\hbar$.

Concluding Remarks

Equation (3.11) does not achieve a complete separation of variables in r and r', since equation (3.12) contains terms in r'. Writing $r=r'$, as one should do, cancels these terms. However, as has been noticed by Hayakawa (1965), all the other terms may be maintained. This might be an indication that Schrödinger's equation would be an approximation of a more general equation, such as equation (3.13).

An estimation of the order of mangitude of the terms of the right-hand side of equation (3.13) for a harmonic oscillator is given in the Appendix below. It is also shown that for the ground-state these terms cancel each other.

To achieve a complete identification of the ensemble of the results obtained in stochastic electrodynamics with wave mechanics one has to

(1) either postulate that $K=\hbar$;
(2) or evaluate K independently of quantum mechanics.

This second alternative, which is in line with the standpoint adopted in the present article, has been considered and will be reported elsewhere.

Appendix

Consider a harmonic oscillator with $V=(m/2)\omega_0{}^2x^2$; Schrödinger's equation is then (Fermi, 1961)

$$\frac{d^2\psi}{dx^2}+\frac{2m}{\hbar^2}\left[E-\frac{m\omega_0{}^2}{2}x^2\right]\psi=0 \tag{A.1}$$

The normalised oscillator eigenfunctions are

$$\psi_n = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{(2^n.n!)}} H_n(\xi) \exp(-\xi^2/2) \tag{A.2}$$

with $\xi = \sqrt{(m\omega_0/\hbar)}.x$; $H_n(\xi)$ is a Hermite polynomial, and

$$E_n = \hbar\omega_0(n + \tfrac{1}{2}) \tag{A.3}$$

For the ground state one has

$$\psi_0 = \left(\frac{m\omega_0}{\hbar\pi}\right)^{1/4} \exp(-\xi^2/2) \tag{A.4}$$

$$E_0 = \frac{\hbar\omega_0}{2} \tag{A.5}$$

To estimate the order of magnitude of the terms of the right-hand side of equation (3.13) one has to compare E_0 to $K\beta$ and to

$$\frac{Sm^2}{K}\cdot\frac{I_\psi(r)}{\psi}$$

Assuming that $K = \hbar$, one has (Surdin, 1970)

$$\frac{\hbar\beta}{E_0} = \frac{2\hbar}{\hbar\omega_0}\cdot\frac{\omega_0{}^2}{\omega_s} = \frac{2\omega_0}{\omega_s} \ll 1 \tag{A.6}$$

according to the assumption made above.

Consider now the expression

$$-\beta\hbar - \frac{Sm^2}{\hbar}\frac{I_\psi(x)}{\psi}$$

which represents the right-hand side of equation (3.13). $I_\psi(x)$ is not known in a closed form; however, since the loss of energy of the oscillator, represented by $-\beta\hbar$, occurs mostly for small values of x, where $\dot{x}$ is maximum, one may restrict the evaluation of $I_\psi(x)$ for small values of x. So that

$$I_\psi(x) \cong -\psi\cdot\frac{\hbar}{m\omega_0}$$

Then

$$-\beta\hbar - \frac{Sm^2}{\hbar}\cdot\frac{I_\psi(x)}{\psi} = -\beta\hbar + \frac{Sm}{\omega_0} \tag{A.7}$$

Since (Surdin, 1970)

$$\frac{Sm}{\omega_0} = \frac{\hbar\omega_0{}^2}{\omega_s} \tag{A.8}$$

equations (A.7) and (A.8) yield

$$-\frac{\omega_0{}^2}{\omega_s}\hbar + \frac{\hbar\omega_0{}^2}{\omega_s} = 0 \tag{A.9}$$

In other words, for a harmonic oscillator in the ground state, the energy radiated by the oscillator is exactly compensated by the energy yielded to the oscillator from the fluctuating electromagnetic field.

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